Supplemental Material for “A unifying implementation of stratum (aka strong) orthogonal arrays” by Ulrike Grömping

1 Construction by Liu and Liu (2015)

Let $\mathbf{V}$ be an $\text{OA}(n, m_{\text{oa}}, s, t)$. The algorithm of Liu and Liu (2015) proceeds as follows: Define a block diagonal $m_{\text{oa}} \times 2k$ matrix $\mathbf{R}$ that has

- k diagonal blocks of identical $b \times 2$ matrices, where $b = t$ for even t and $b = t + 1$ for odd t,
- followed by $q = m_{\text{oa}} - bk$ rows of zeroes (none if $q = 0$).

The design $\mathbf{D}$ is obtained as $\mathbf{D} = \mathbf{VR}$; remember that Liu and Liu denoted the levels in $\mathbf{V}$ by $-(s-1), -(s-3), \ldots, +(s-1)$. Each column of the $b \times 2$ matrix holds exactly one element of $s^0 = 1, s^1 = s, \ldots, s^{t-1}$ (where the list stops at s for $s = 2$), with an additional zero element for odd t; these values carry a positive or negative sign. It is thus straightforward, if a little bit tedious, to define $\mathbf{A}$, $\mathbf{B}$ etc. according to the following rule:

- $\mathbf{a}_\ell$ is obtained from the unique column $\mathbf{v}_j$ of $\mathbf{V}$ for which column $\mathbf{r}_\ell$ holds the entry $\pm s^{t-1}$,
- $\mathbf{b}_\ell$ is obtained from the unique column $\mathbf{v}_j$ of $\mathbf{V}$ for which column $\mathbf{r}_\ell$ holds the entry $\pm s^{t-2}$,
- and so forth.

Where the entry in the matrix $\mathbf{R}$ is positive, $\mathbf{v}_j$ is used directly; where the entry in the matrix $\mathbf{R}$ has a minus sign, $s - 1 - \mathbf{v}_j$ is used (reversal of levels). Equations (6) to (9) gave the results of these allocations for $t = 2$ to $t = 4$. The matrix constructions behind these equations are detailed below:

For a strength 2 $\text{OA}(n, m, s, 2)$ called $\mathbf{V}$, the $2\lfloor m/2 \rfloor$ columns of the matrices $\mathbf{A}$ and $\mathbf{B}$ are obtained as follows:

$$\mathbf{a}_\ell = \begin{cases} \mathbf{v}_{\ell+1} & \ell \text{ odd} \\ \mathbf{v}_{\ell-1} & \ell \text{ even} \end{cases}, \qquad \mathbf{b}_\ell = \begin{cases} \mathbf{v}_\ell = \mathbf{a}_{\ell+1} & \ell \text{ odd} \\ s - 1 - \mathbf{v}_\ell = s - 1 - \mathbf{a}_{\ell-1} & \ell \text{ even} \end{cases}, \quad \ell = 1, \ldots, 2\lfloor m/2 \rfloor,$$

where $1 \leq j \leq \lfloor m/2 \rfloor$.

For a strength 3 $\text{OA}(n, m, s, 3)$ called $\mathbf{V}$, the $2\lfloor m/4 \rfloor$ columns of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are obtained as follows:

$$\mathbf{a}_\ell = \begin{cases} \mathbf{v}_{2\ell+1} & \ell \text{ odd} \\ \mathbf{v}_{2\ell-3} & \ell \text{ even} \end{cases}, \quad \mathbf{b}_\ell = \mathbf{v}_{2\ell}, \quad \mathbf{c}_\ell = \begin{cases} \mathbf{v}_{2\ell-1} = \mathbf{a}_{\ell+1} & \ell \text{ odd} \\ s - 1 - \mathbf{v}_{2\ell-1} = s - 1 - \mathbf{a}_{\ell-1} & \ell \text{ even} \end{cases}, \quad \ell = 1, \ldots, 2\lfloor m/4 \rfloor.$$

If $m - 4\lfloor m/4 \rfloor = 3$, an additional column can be added as follows:

$$\mathbf{a}_{2\lfloor m/4 \rfloor + 1} = \mathbf{v}_m, \quad \mathbf{b}_{2\lfloor m/4 \rfloor + 1} = \mathbf{v}_{m-1}, \quad \mathbf{c}_{2\lfloor m/4 \rfloor + 1} = \mathbf{v}_{m-2}.$$

For a strength 4 $\text{OA}(n, m, s, 4)$ called $\mathbf{V}$, the $2\lfloor m/4 \rfloor$ columns of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ are obtained as follows ($\ell = 1, \ldots, \lfloor m/4 \rfloor$): For odd ℓ,

$$\mathbf{a}_{1;\ell} = \mathbf{v}_{2\ell+2}, \quad \mathbf{a}_{2;\ell} = \mathbf{v}_{2\ell+1}, \quad \mathbf{a}_{3;\ell} = \mathbf{v}_{2\ell} = \mathbf{b}_{\ell+1}, \quad \mathbf{a}_{4;\ell} = \mathbf{v}_{2\ell-1} = \mathbf{a}_{\ell+1},$$

for even ℓ,

$$\mathbf{a}_{1;\ell} = \mathbf{v}_{2\ell-3}, \quad \mathbf{a}_{2;\ell} = \mathbf{v}_{2\ell-2}, \quad \mathbf{a}_{3;\ell} = s - 1 - \mathbf{v}_{2\ell-1} = s - 1 - \mathbf{b}_{\ell-1}, \quad \mathbf{a}_{4;\ell} = s - 1 - \mathbf{v}_{2\ell} = s - 1 - \mathbf{a}_{\ell-1}.$$

2 Proof of Proposition 1

Let $\mathbf{A}$ and $\mathbf{B}$ be OA$(n, m, s, 2)$, and let $\mathbf{A}^*$ and $\mathbf{B}^*$ denote those matrices after subtracting $(s-1)/2$ (i.e., centered versions of the matrices). Li et al.'s (2021) algorithm proceeds as follows:

a) Obtain an $n \times 2m'$ array $\mathbf{C} = (\mathbf{C}_1, \ldots, \mathbf{C}_{m'/2})$ by interleaving the columns of $\mathbf{A}$ and $\mathbf{B}$ as follows:

$$\mathbf{C}_\ell = (\mathbf{a}_{2\ell-1}, \mathbf{b}_{2\ell-1}, \mathbf{a}_{2\ell}, \mathbf{b}_{2\ell}), \quad \ell = 1, \ldots, m'/2.$$

b) Obtain the column-centered matrix $\mathbf{C}^*$ by subtracting $(s-1)/2$ from each element of $\mathbf{C}$, so that elements are in the interval $[-(s-1)/2, (s-1)/2]$, i.e., $\mathbf{C}^*$ interleaves $\mathbf{A}^*$ and $\mathbf{B}^*$.

c) Obtain $n \times 2$ matrices $\mathbf{D}_\ell^* = C_\ell^* \mathbf{V}$, with

$$\mathbf{V} = \begin{pmatrix} s^2 & s & 0 & 1 \\ -1 & 0 & s^2 & s \end{pmatrix}^\top.$$

d) Obtain the $n \times m'$ design matrix

$$\mathbf{D} = (\mathbf{D}_1^*, \ldots, \mathbf{D}_{m'/2}^*) + (s^3 - 1)/2.$$

The first column of $\mathbf{D}_\ell^*$ is the $2\ell - 1^{th}$ column of $\mathbf{D}^*$,

$$\mathbf{d}_{2\ell-1}^* = s^2 \mathbf{a}_{2\ell-1}^* + s\mathbf{b}_{2\ell-1}^* + \mathbf{a}_{2\ell}^*,$$

the second column is

$$\mathbf{d}_{2\ell}^* = s^2 \mathbf{a}_{2\ell}^* + s\mathbf{b}_{2\ell}^* - \mathbf{a}_{2\ell-1}^*.$$

Clearly, $\mathbf{D}^* = s^2\mathbf{A}^* + s\mathbf{B}^* + \mathbf{C}^*$ with the columns of $\mathbf{C}^*$ obtained from $\mathbf{A}^*$. Now, observe that the superscript * stands for subtraction of a constant only; the only position in which this matters is the subtraction of $\mathbf{a}_{2\ell-1}^*$, for which the "$-$" after subtraction of the center value corresponds to a reversal of the levels, which can also be written as $s - 1 - \mathbf{a}_{2\ell-1}$ for the original coding $0, \ldots, s-1$.

3 Example constructions for Shi and Tang Families 2 and 3

Section 4.2 provided the recursive construction for Families 2 and 3. It will be applied to two examples in this appendix.

Example: Constructing an SOA$(64, 16, 8, 3)$ or an OSOA$(64, 15, 8, 3+)$

$64 = 2^6$, i.e., $k = 6$ is even. We need a matrix $\mathbf{Y}$ with $2^{k-2} = 16$ rows in order to obtain matrices $\mathbf{A}$ and $\mathbf{B}$ with 2^k rows. We already saw in Section 4.2 that $Y_\mathbf{Y} = c(2, 3, 1, 8, 10, 11, 9, 12, 14, 15, 13, 4, 6, 7, 5)$, which arises from applying Proposition 3 to the start vector $Y_{\mathbf{Y}_2} = 231$ (with $k = 2$ in the proposition). Corollary 1 tells us that $\mathbf{A}$ holds Yates matrix columns 33 to 47 (in that order) and $\mathbf{B}$ holds Yates matrix columns $16 + Y_\mathbf{Y}$, and Lemma 15 tells us to use Yates matrix columns 1 to 15 for obtaining the OSOA with $n/4 - 1 = 15$ columns. For obtaining the SOA with 16 columns, one can add Yates column 32 to $\mathbf{A}$, Yates column 16 to $\mathbf{B}$, and an arbitrary column from Yates columns 1 to 15 to matrix $\mathbf{C}$, so that matrix $\mathbf{C}$ has one duplicate column pair. According to Lemma 15, this implies orthogonality for most column pairs, with the exception of a non-zero correlation for the pair that have the same $\mathbf{C}$ matrix column.

Example: Constructing Family 2 and Family 3 designs in 32 and 128 runs (k odd)

The start values for a design in 2^5 runs ($k = 5$) have $2^{5-2} - 1$ elements and were given in Lemma 13 as $Y_\mathbf{X} = 1234567$, $Y_\mathbf{Y} = 7521643$, and $Y_\mathbf{Z} = 6715324$.

The resulting start columns for matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A} +_2 \mathbf{B}$ are given as
$Y_{\mathbf{A}} = (16, 17, 18, 19, 20, 21, 22, 23)$,
$Y_{\mathbf{B}} = (8, 15, 13, 10, 9, 14, 12, 11)$,
$Y_{\mathbf{A}+_2\mathbf{B}} = (24, 30, 31, 25, 29, 27, 26, 28)$,
for an $\mathrm{SOA}(32, 8, 8, 3)$ with properties α and β.

According to Lemma 15, eight corresponding columns for $\mathbf{C}$ should be obtained from Yates columns 1 to 7, with one duplicate, and the resulting array has a non-zero correlation for the pair of columns that share the same $\mathbf{C}$ column. If one only needs seven columns, omitting the first columns from $\mathbf{A}$ and $\mathbf{B}$ and using Yates columns 1 to 7 for $\mathbf{C}$ yields an $\mathrm{OSOA}(32, 7, 8, 3+)$, since the array is from Shi and Tang's Family 3 and additionally fulfills all requirements of Lemma 5.

One step of the recursion yields

$Y_{\mathbf{X}} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31)$,
$Y_{\mathbf{Y}} = (7, 5, 2, 1, 6, 4, 3, 16, 23, 21, 18, 17, 22, 20, 19, 24, 31, 29, 26, 25, 30, 28, 27, 8, 15, 13, 10, 9, 14, 12, 11)$,
$Y_{\mathbf{Z}} = (6, 7, 1, 5, 3, 2, 4, 24, 30, 31, 25, 29, 27, 26, 28, 8, 14, 15, 9, 13, 11, 10, 12, 16, 22, 23, 17, 21, 19, 18, 20)$
for the construction of an $\mathrm{OSOA}(128, 31, 8, 3+)$, whose Yates matrix columns are
$Y_{\mathbf{A}} = (65, \ldots, 95)$,
$Y_{\mathbf{B}} = (39, 37, 34, 33, 38, 36, 35, 48, 55, 53, 50, 49, 54, 52, 51, 56, 63, 61, 58, 57, 62, 60, 59, 40, 47, 45, 42, 41, 46, 44, 43)$,
$Y_{\mathbf{A}+_2\mathbf{B}} = (102, 103, 97, 101, 99, 98, 100, 120, 126, 127, 121, 125, 123, 122, 124, 104, 110, 111, 105, 109, 107, 106, 108, 112, 118, 119, 113, 117, 115, 114, 116)$.

Orthogonal columns are guaranteed by choosing Yates columns 1 to 31 for matrix $\mathbf{C}$. The analogous construction of the Family 2 $\mathrm{SOA}(128, 32, 8, 3)$ with properties α and β additionally uses Yates columns 64 and 32 as the first columns of matrices $\mathbf{A}$ and $\mathbf{B}$, and adds another column from 1 to 31 to matrix $\mathbf{C}$.